Universal Error-Reducing Methodology on Option Pricing

Hua-Yi Lin

Advisor: Tian-Shyr Dai

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Agenda

Literature Review:

P.Carr and D.B.Madan (1999). Option valuation using the fast Fourier transform. *Journal of Computational Finance.*

C.Y.Chiu and H.Y.Lin (2011). Universal error-reducing methodology on option pricing.

Option valuation using the fast Fourier transform

We would like to use a model which is supported in a general equilibrium and which is capable of removing the biases of the standard Black-Scholes model.

In many models the characteristic function is simple and known analytically, while the density function is complicated.(e.g. Lévy processes)

Theorem (Lévy-Khintchine representation)

Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ with characteristic triplet (A, ν, γ)

Then $E[e^{iz.X_t}] = e^{t\phi(z)}$, $z \in \mathbb{R}^d$

with $\phi(z) = -\frac{1}{2}z.Az + i\gamma.z + \int_{\mathbb{R}^d} e^{iz.x} - 1 - iz.x\,1_{\{|x|\leq 1\}}\,\nu(dx)$

Option valuation using the fast Fourier transform

$s_T := \ln(S_T) \qquad k := \ln(K)$

$q_T(s)$: risk-neutral density of s_T

$\phi_T(u) := \int_{-\infty}^{\infty} e^{ius} q_T(s)\, ds$: characteristic function of $q_T(s)$

$C_T(k)$: T-maturity call option with strike e^k

$C_T(k) := \int_k^{\infty} e^{-rT}(e^s - e^k) q_T(s)\, ds$

$\because C_T(k) \to S_0$ as $k \to -\infty \qquad \therefore C_T(k) \notin L^2(\mathbb{R})$

Goal: choose $\alpha > 0$ s.t. $c_T(k) := e^{\alpha k} C_T(k) \in L^2(\mathbb{R})$

$\psi_T(v) := \int_{-\infty}^{\infty} e^{ivk} c_T(k)\, dk$: characteristic function of $c_T(k)$

Option valuation using the fast Fourier transform

$$\therefore C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi_T(v)\, dv = \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-ivk} \psi_T(v)\, dv$$

$$\psi_T(v) = \int_{-\infty}^{\infty} e^{ivk} c_T(k)\, dk = \int_{-\infty}^{\infty} e^{ivk} e^{\alpha k} C_T(k)\, dk$$

$$= \int_{-\infty}^{\infty} e^{ivk} e^{\alpha k} e^{-rT} \int_{k}^{\infty} (e^s - e^k) q_T(s)\, ds\, dk$$

$$= e^{-rT} \int_{-\infty}^{\infty} q_T(s) \int_{-\infty}^{s} e^{ivk} e^{\alpha k} (e^s - e^k)\, dk\, ds$$

$$= e^{-rT} \int_{-\infty}^{\infty} q_T(s) \int_{-\infty}^{s} e^{k(iv+\alpha)+s} - e^{k(iv+\alpha+1)}\, dk\, ds$$

$$= e^{-rT} \int_{-\infty}^{\infty} q_T(s) \left[\frac{e^{k(iv+\alpha)+s}}{iv+\alpha} - \frac{e^{k(iv+\alpha+1)}}{iv+\alpha+1} \right]_{-\infty}^{s} ds$$

$$= e^{-rT} \int_{-\infty}^{\infty} q_T(s) \frac{e^{s(iv+\alpha+1)}}{(iv+\alpha)(iv+\alpha+1)}\, ds$$

$$= \frac{e^{-rT}}{(iv+\alpha)(iv+\alpha+1)} \int_{-\infty}^{\infty} q_T(s)\, e^{is(v-i\alpha-i)}\, ds = \frac{e^{-rT} \phi_T(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}$$

Option valuation using the fast Fourier transform

Note: $Re[\psi_T(v)] = Re[\psi_T(-v)]$, $Im[\psi_T(v)] = Im[\psi_T(-v)]$

$\psi_T(v) = \int_{-\infty}^{\infty} e^{ivk} e^{\alpha k} C_T(k)\, dk$

$= \int_{-\infty}^{\infty} [\cos(vk) + i \sin(vk)] e^{\alpha k} C_T(k)\, dk$

$= \int_{-\infty}^{\infty} \cos(vk) e^{\alpha k} C_T(k)\, dk + i \int_{-\infty}^{\infty} \sin(vk) e^{\alpha k} C_T(k)\, dk$

Note: upper bound on α : $E[S_T^{\alpha+1}] < \infty$

$\psi_T(0) < \infty \Rightarrow e^{\alpha k} C_T(k) \in L^1(\mathbb{R}) \Rightarrow e^{\alpha k} C_T(k) \in L^2(\mathbb{R})$

$\phi_T(-(\alpha+1)i) = E[S_T^{\alpha+1}] < \infty \Rightarrow \psi_T(0) = \frac{e^{-rT}\phi_T(-(\alpha+1)i)}{\alpha^2+\alpha} < \infty$

Option valuation using the fast Fourier transform

The FFT is an efficient algorithm for computing the sum

$$w(k) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)}\chi(j) \quad for\ k = 1 \ldots N$$

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi} \int_0^{N\eta} e^{-ivk}\psi_T(v)\,dv$$

$$\approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-i\eta(j-1)k}\psi_T(\eta(j-1))\eta := \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-iv_j k}\psi_T(v_j)\eta$$

$$k_u := -b + \lambda(u-1) \quad u = 1 \ldots N \qquad \lambda = \frac{2\pi}{\eta N} \qquad b = \frac{N\lambda}{2}$$

$$C_T(k_u) \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-iv_j(-b+\lambda(u-1))}\psi_T(v_j)\eta$$

$$= \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-i\lambda\eta(j-1)(u-1)} e^{ibv_j}\psi_T(v_j)\eta$$

$$= \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)} e^{ibv_j}\psi_T(v_j)\eta$$

Universal error-reducing methodology on option pricing

$$\psi_T(v) = \int_{-\infty}^{\infty} e^{ivk} e^{\alpha k} e^{-rT} \int_k^{\infty} (e^s - e^k) q_T(s)\, ds\, dk$$

$$= \int_{-\infty}^{\infty} e^{ivk} e^{\alpha k} e^{-rT} \int_k^{\infty} (e^s - e^k) q_T^{proxy}(s)\, ds\, dk$$

$$+ \int_{-\infty}^{\infty} e^{ivk} e^{\alpha k} e^{-rT} \int_k^{\infty} (e^s - e^k) q_T^{residue}(s)\, ds\, dk$$

$$= \psi_T^{proxy}(v) + \psi_T^{residue}(v)$$

$$\therefore C_T(k) = \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{-ivk} \psi_T(v)\, dv$$

$$= \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{-ivk} [\psi_T^{proxy}(v) + \psi_T^{residue}(v)]\, dv$$

$$= C_T^{proxy}(k) + \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{-ivk} \psi_T^{residue}(v)\, dv$$